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APPROXIMATING MULTIPLE ÎTO INTEGRALS
WITH "BAND LIMITED" PROCESSES

by

Harold J. Kushner and Hai Huang

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Approximating Multiple Îto Integrals

With "Band Limited" Processes

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ABSTRACT

Let $n^\epsilon(\cdot)$ denote a "wide-band width" vector valued process. The paper is concerned with limits of $\int_0^t \int_0^\tau n^{\epsilon'}(\tau) L(\tau,s) n^\epsilon(s) d\tau ds$, and for the m-multiple integral case. For the most important case, a weak convergence result is obtained, and the "correction" terms exhibited. The method is such that the conditions used can readily be weakened. An application to a likelihood functional and hypothesis testing problem is given. There, the weak convergence result (rather than mere convergence of finite dimensional distributions) is essential if the limit approximation is to make sense as an approximation to the likelihood functional. The correction terms depend only on the limit (as *epsilon approaches limit of* $\epsilon \to 0$) of the correlation function of the (renormalized) $n^\epsilon(\cdot)$. *n sub epsilon*

1. Introduction. We are concerned with the set of problems introduced by Balakrishnan in [1]. Let the matrix valued kernel $L(\cdot,\cdot)$ be such that $\int_0^T\int_0^T |L(t,s)|^2 ds\,dt < \infty$ for each $T < \infty$, and $L(t,s) = 0$ for $s > t$. For each fixed T, define the operator L on $L_2[0,T]$ by $Lf(t) = \int_0^t L(t,s)f(s)ds$. Then the adjoint operator L^* is defined by $L^*f(t) = \int_t^T L'(s,t)f(s)ds$ and $(L + L^*)f(t) = \int_0^T (L(t,s) + L'(s,t))f(s)ds$. Let $(L + L^*)$ be nuclear [2] for each $T < \infty$. Then [2] for each $T < \infty$, there are λ_i, M_i, K_i such that $\sum|\lambda_i| < \infty$, $||M_i|| \leq 1$, $||K_i|| \leq 1$ ($||\ ||$ is the $L_2[0,T]$ norm) and

(1.1a)
$$L(t,s) + L'(s,t) = \sum_1^\infty \lambda_i M_i(t) K_i(s)\ ,\ 0 \leq s,t \leq T.$$

Hence for $T \geq t \geq s \geq 0$,

(1.1b)
$$L(t,s) = \sum_1^\infty \lambda_i M_i(t) K_i(s).$$

Without loss of generality, assume that all M_i, K_i, L are $r \times r$ matrices.

If $X^n(\cdot)$ is a sequence of random processes (with paths in some function space), we write $X^n(\cdot) \Rightarrow X(\cdot)$ to denote weak convergence [3],[4] to a process $X(\cdot)$. Let $n^\epsilon(\cdot)$ denote a "wide-band" noise process such that as $\epsilon \to 0$ the process defined by

(1.2)
$$\int_0^t n^\epsilon(s)ds = N^\epsilon(t)$$

converges weakly to a standard Wiener process $w(\cdot)$. The problem is to find the weak limits of repeated integrals such as

(1.3)
$$x^\epsilon(t) = \int_0^t d\tau \int_0^\tau ds\ n^\epsilon(\tau)'L(\tau,s)n^\epsilon(s).$$

We also treat the case of m-repeated integrals.

In [1], $n^\varepsilon(\cdot)$ was Gaussian and it was obtained from a Wiener process in a very particular way. Also, convergence (in the mean square sense) was proved only for each fixed t. In this paper, the problem is treated in the context of the theory of weak convergence of measures [3],[4]. Most of the results (including those for the important application of Section 5) are of the weak convergence type. But some are a combintation of weak convergence and convergence of finite dimensional distributions. A much broader class of noise processes can be dealt with. This is important in applications, since we should not have to require that the actual physical noise is the particular smoothed functional of $w(\cdot)$ used in [1]. (In [1], the $n^\varepsilon(\cdot)$ was such that it's spectral density was 1 on some interval $[-M_\varepsilon, M_\varepsilon]$, and zero outside, where $M_\varepsilon \to \infty$ as $\varepsilon \to 0$.)

Owing to the nature of weak convergence, a broad class of path functionals (including passage times, a particularly important case in applications) of the process $x^\varepsilon(\cdot)$ converges in distribution to the same functional of the limit process. As in [1], we apply the result to the problem of approximating a likelihood functional (Section 5). This problem demonstrates the importance of the weak convergence (rather than convergence in distribution for each t). The weak convergence method is essential, if the results of the approximation are to be used for a sequential hypothesis test, and it is still important even if the test is to be conducted at a fixed time only. Further comments on this and related points appear in Section 5.

In the next section, the noise model is discussed, and some definitions collected in Section 3. In Section 4, the weak convergence result is proved first (for the double integral case) when the sum in (1.1b) is finite and the H_i, K_i are continuous. Other results will follow from this. The likelihood ratio problem appears in Section 5, and Section 6 outlines the method for the multiple integral. The results here can also be applied to the non-linear filtering problem dealt with by Ocone [5], where the filter is represented as a sum of multivariate Ito integrals, but where the observation noise is "wide-band". It should be clear that the method is <u>equally applicable</u> to the case where (1.3) is replaced by $\int_0^t d\tau \int_0^\tau ds \, n_2^\varepsilon(\tau)'L(\tau,s)n_1^\varepsilon(s)$, and $n_2^\varepsilon(\cdot)$ is not the same as $n_1^\varepsilon(\cdot)$ (and also for the analogous m-multiple integral), but for simplicity of notation, we stick to the simpler case.

The correction terms depend only on the correlation funtion of the $y(\cdot)$ process introduced in Section 2, and thus are robust with respect to the underlying noise model. Although, it will not be pursued, one can use our method to deal with cases where the kernel $L(\cdot,\cdot)$ depends on $n^\varepsilon(\cdot)$ also. Such results are an additional advantage of the weak convergence approach.

2. The Noise Model. Suppose that $y(\cdot)$ is a zero mean stationary process and define $y^\varepsilon(t) = y(t/\varepsilon^2)$. If $y(\cdot)$ has spectral density $S(\omega)$, then $y^\varepsilon(\cdot)/\varepsilon$ has spectral density $S(\varepsilon^2\omega)$. To obtain the $n^\varepsilon(\cdot)$ used in [1], let $y(\cdot)$ be a (scalar valued) zero mean Gaussian process with spectral density 1 on $[-M,M]$ and zero elsewhere, and use $y^\varepsilon(\cdot)/\varepsilon = n^\varepsilon(\cdot)$. Note that $R(\tau) =$ $= Ey(\tau)y(0)$ satisfies

$$R(\tau) = \frac{1}{2\pi} \int_{-M}^{M} e^{i\omega\tau} d\omega = \sin M\tau / \pi\tau$$

$$\int_{-\infty}^{\infty} R(\tau) d\tau = 1.$$

The form $y^\varepsilon(\cdot)/\varepsilon = n^\varepsilon(\cdot)$ is a common and useful way of obtaining a "wide-band" process. The method used here is not restricted to the use of such a form. In order to simplify the use of published results, we proceed as follows in the theorems. We let $n^\varepsilon(\cdot) = y^\varepsilon(\cdot)/\varepsilon$, where $y(\cdot)$ is a right continuous (vector valued) process satisfying either (A1.a) or (A1.b) below. After the proof of Theorem 1, we state two sets of more general conditions under which the results hold. In general $y(\cdot)$ need not be stationary. One particular interesting case which fits the conditions described after Theorem 1 is (under appropriate assumptions on $h_\varepsilon(\cdot)$ and the Poisson jump process $N_\varepsilon(\cdot)$)

$$n^\varepsilon(t) = \int_{-\infty}^{t} h_\varepsilon(t-s)\,dN_\varepsilon(s).$$

A1. Either (a): $y(\cdot)$ is stationary, mean zero and Gaussian with a rational spectral density function, or (b): it is stationary, bounded and strongly ϕ-mixing[†] [3] with mixing rate $\phi(\cdot)$ satisfying $\int_0^\infty \phi^{\frac{1}{2}}(u)\,du < \infty$.

Under (A1.a), there is a stable matrix A_1, matrices B_1 and H_1 and a standard Wiener process $w_1(\cdot)$ such that

(2.1) $y = H_1 Y_1$, where $dY_1 = A_1 Y_1 dt + B_1 dw_1$,

We normalize such that Covar $Y_1(t) = I$. Define $R_1(t) = EY_1(t)Y_1'(0)$ (stationary value). Then $R_1(t) \to 0$ exponentially and for $s > t$

(2.2a) $E[y(s)|Y_1(u), u \le t] = H_1 R_1(s-t) Y_1(t).$

We also use the property that if $\xi_1,\ldots,\xi_4$ are zero mean and jointly Gaussian,

---

[†] $y(\cdot)$ is strongly ϕ-mixing if there is a function $\phi(s)$ which goes to zero $s \to \infty$ such that for each t and measurable set A depending only on $y(u), u \le t$, and measurable set B depending only on $y(u), u \ge t+s$, $|P(B|A) - P(B)| \le \phi(s)$.

then

(2.2b) $E\xi_1\xi_2\xi_3\xi_4 = E\xi_1\xi_2 E\xi_3\xi_4 + E\xi_1\xi_3 E\xi_2\xi_4 + E\xi_1\xi_4 E\xi_2\xi_3.$

Assume (A1.b), let $\tau \geq 0$, $s \geq 0$ and let $y(t)$ be bounded by k_1. Then [6]

$$\left| E[y(t+s)|y(u), u \leq t] \right| \leq 2k_1\phi(s)$$

$$\left| Ey(t+s)y'(t+s+\tau) \right| \leq 2k_1^2\phi(\tau)$$

(2.3a)

$$E \equiv \left| E[y(t+s)y'(t+s+\tau)|y(u), u \leq t] - Ey(t+s)y'(t+s+\tau) \right|$$

$$\leq 4k_1^2\phi(s)$$

and

$$E \leq 4k_1^2\phi(\tau).$$

Hence $E \leq 4k_1^2\phi^{\frac{1}{2}}(s)\phi^{\frac{1}{2}}(\tau)$.

Let $s_1 < s_2 < s_3 < s_4$, and let $y_i(\cdot)$ denote the scalar components of $y(\cdot)$. Then the mixing (A1.b) and $Ey_i(s_1)Ey_j(s_2)y_n(s_3)y_\ell(s_4)=0=Ey_i(s_1)y_j(s_2)y_n(s_3)Ey_\ell(s_4)$ imply that

$$E_1 = \left| Ey_i(s_1)y_j(s_2)y_n(s_3)y_\ell(s_4) \right| \leq k\phi(s_4-s_3)$$

$$E_1 \leq k\phi(s_2-s_1)$$

Hence $E_1 \leq k\phi^{\frac{1}{2}}(s_4-s_3)\phi^{\frac{1}{2}}(s_2-s_1)$.

Here and in the sequel k denotes a constant whose value might change from case to case. Both (2.2) and (2.3) will be useful for evaluating various integrals in the following theorems, and are generally used without specific mention.

3. **Some Definitions.** We work with the space $D^r[0,\infty)$ of R^r-valued right continuous functions with left hand limits, and the Skorohod topology [3],[4]. Let $\mathscr{F}_t$ and $\mathscr{F}_t^\epsilon$ denote, respectively, the minimal (completed) σ-algebras over which $\{y(s), s \leq t\}$ and $\{y^\epsilon(s), s \leq t\}$ are measurable, and let E_t and E_t^ϵ denote the corresponding conditional expectation operators. Following Rishel [7] and Kurtz [8], define p-lim and $\hat{A}^\epsilon$ as follows. We

say p-lim $f_n = 0$ if for each $T < \infty$, $\sup_{n,t \leq T} E|f_n(t)| < \infty$ and $E|f_n(t)| \to 0$ for each t. $f(\cdot)$ is p-right continuous if p-lim$_{\delta \downarrow 0}[f(\cdot+\delta) - f(\cdot)] = 0$. Let f be progressively measurable with respect to the family $\{\mathscr{F}_t^\epsilon, t < \infty\}$. Then we say that $f \in \mathscr{D}(\hat{A}^\epsilon)$ and $\hat{A}^\epsilon f = g$ if

$$\text{p-lim} \ \frac{\hat{E}_t^\epsilon f(t+\delta) - f(t)}{\delta} - g(t) = 0$$

and g is p-right continuous. If $f \in (\hat{A}^\epsilon)$, then

(3.1)
$$M_f(t) = f(t) - f(0) - \int_0^t \hat{A}^\epsilon f(s)\,ds$$

is an $\mathscr{F}_t^\epsilon$-martingale [8].

4. The Convergence Theorem for Double Integrals. In preparation for the sequel, consider the case where $L(t,s) = M(t)K(s)$, where $M(\cdot) = M_1(\cdot)$ and $K(\cdot) = K_1(\cdot)$ are continuous. Define

(4.1)
$$z_1^\epsilon(t) = \int_0^t K(s)n^\epsilon(s)\,ds$$

$$x_1^\epsilon(t) = \int_0^t n^\epsilon(\tau)'M(\tau)z_1^\epsilon(\tau)\,d\tau.$$

Then $x_1^\epsilon(t) = x^\epsilon(t)$. For $L(t,s) = \sum_1^m M_i(t)K_i(s)$, where $M_i(\cdot)$ and $K_i(\cdot)$ are continuous, define $z^\epsilon(\cdot) = (z_1^\epsilon(\cdot),\ldots,z_m^\epsilon(\cdot))'$ and

(4.2)
$$z_i^\epsilon(t) = \int_0^t K_i(s)n^\epsilon(s)\,ds,$$

$$x_i^\epsilon(t) = \int_0^t n^\epsilon(\tau)'M_i(\tau)z_i^\epsilon(\tau)\,d\tau, \ i \leq m.$$

Then

(4.3)
$$x^\epsilon(t) = \sum_1^m x_i^\epsilon(t).$$

Limits of the above sequences are dealt with in Theorem 1. Define $\overline{R} =$

$= \int_{-\infty}^{\infty} R(u)\,du$, where $R(u) = Ey(u)y'(0)$ and $R_{ij}^{+} = \int_{0}^{\infty} Ey_i(s)y_j(0)\,ds$. Let $\sqrt{\bar{R}}$ denote the "positive" square root of the matrix $\bar{R}$ and note (for future use) that $R(u) = R'(-u)$.

Theorem 1. _Assume_ (A1) _and let_ $L(t,s)$ _take the form_ $\sum_{1}^{m} M_i(t)K_i(s)$, _where_ M_i _and_ K_i _are continuous. Then_ $\{z_i^\epsilon(\cdot),\ i \le m,\ x^\epsilon(\cdot)\} = \{X^\epsilon(\cdot)\}$ _is tight in_ $D^{mr+1}[0,\infty)$ _and converges weakly to the process_ $\{z_i(\cdot),\ i \le m,\ x(\cdot)\} =$ $= X(\cdot)$, _where for_ $m = 1$ (_writing_ $z=z_1$, $M=M_1$, $K=K_1$ _here_)

$$
(4.4) \qquad
\begin{aligned}
dz &= 0 + \\
dx &= \sum_{i,j} R_{ij}^{+}\, L_{ij}(t,t)\,dt + \begin{pmatrix} K \\ z'M' \end{pmatrix} \sqrt{\bar{R}}\; dw,
\end{aligned}
$$

where $L_{ij}(t,t)$ _is the_ ij^{th} _component of_ $L(t,t)$, _and_ $w(\cdot)$ _is a standard Wiener process. For general_ m,

$$
(4.5a) \qquad \begin{pmatrix} dz_1 \\ \cdot \\ \cdot \\ \cdot \\ dz_m \end{pmatrix} = \begin{pmatrix} K_1(t) \\ \cdot \\ \cdot \\ \cdot \\ K_m(t) \end{pmatrix} \sqrt{\bar{R}}\; dw
$$

$$
(4.5b) \qquad dx = \sum_{i,j} R_{ij}^{+} L_{ij}(t,t)\,dt + \sum_{i} z_i' M_i'(t)\sqrt{\bar{R}}\; dw
$$

Also

$$
(4.5c) \qquad \int_{0}^{\cdot} y(s)\,ds/\epsilon \to \sqrt{\bar{R}}\; w(\cdot).
$$

The component (4.5b) equals

$$
(4.6)^{\dagger} \quad x(t) = \int_{0}^{t} \sum_{i,j} R_{ij}^{+} L_{ij}(s,s)\,ds + \int_{0}^{t}\int_{0}^{\tau} (\sqrt{\bar{R}}\; dw(\tau))'L(\tau,s)\sqrt{\bar{R}}\; dw(s).
$$

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† With the noise model of [1], $R_{ij}^{+} = \delta_{ij}/\alpha$, in which case the correction term is $\frac{1}{2}\int \sum L_{ii}(t,t)\,dt$, which is consistent with the results in [1].

Proof. Under the conditons on the noise, and the continuity of K_i and M_i, the result follows by the perturbed test function method of [9],[10], and we only show how to identify the terms of the limit operator and verify the assumptions in [10]. Reference [10] uses "truncated" processes, but we ignore this in what follows, since we are only interested in identifying the terms and assumptions used in [10]. The method is simple here, since the equation for the z_i^ϵ is not "feedback", and the equation for the x_i^ϵ does not depend on the $\{x_j^\epsilon, j \leq m\}$, but only on $z_i^\epsilon(\cdot)$ and $y^\epsilon(\cdot)$. For simplicity, until further notice, we do the special case where $L(t,s) = M(t)K(s)$ (i.e., $m = 1$). Let the test function $f(\cdot)$ have compact support and continuous mixed second partial derivatives. Then (write $f(t) = f(x^\epsilon(t), z^\epsilon(t))$ and $X = (x,z), X^\epsilon(t) = (x^\epsilon(t), z^\epsilon(t))$

$$\hat{A}^\epsilon f(t) = f'_z(X^\epsilon(t))\dot{z}^\epsilon(t) + f_x(X^\epsilon(t))\dot{x}^\epsilon(t)$$

$$= f'_z(X^\epsilon(t))K(t)y^\epsilon(t)/\epsilon + f_x(X^\epsilon(t))y^{\epsilon'}(t)M(t)z^\epsilon(t)/\epsilon.$$

Fix T. We define the test function perturbations as follows (see [10] for more detail). Define $f_1^\epsilon(t) = f_1^\epsilon(X^\epsilon(t),t)$, where

$$f_1^\epsilon(X,t) = \frac{1}{\epsilon}\int_t^T d\tau \, f'_z(X) \, K(\tau)E_t^\epsilon y^\epsilon(\tau)$$

$$+ \frac{1}{\epsilon}\int_t^T d\tau \, f_x(X) \, E_t^\epsilon y^{\epsilon'}(\tau)M(\tau)z \quad \equiv \frac{1}{\epsilon}\int_t^T E_t^\epsilon G(X,\tau)d\tau$$

Then $f_1^\epsilon(\cdot) \in \mathscr{D}(\hat{A}^\epsilon)$ and

$$\hat{A}^\epsilon f_1^\epsilon(t) = -G(X^\epsilon(t),t)/\epsilon + \frac{1}{\epsilon}\int_t^T E_t^\epsilon(\frac{d}{dt} G(X^\epsilon(t),\tau))d\tau.$$

Note that, by a change of variables $s/\epsilon^2 \to s$, we get

$$f_1^\epsilon(t) = \epsilon\int_{t/\epsilon^2}^{T/\epsilon^2} d\tau \, f_x(X)E_t^\epsilon y'(\tau/\epsilon^2)M(\epsilon^2\tau)z.$$

which is $0(\epsilon)$ in the mixing case and $0(|y_1(t)|\epsilon)$ in the Gaussian case. Such a transformation will be used frequently, without specific mention.

We have (with $X^\varepsilon = X^\varepsilon(t)$)

$$\hat{A}^\varepsilon f_1^\varepsilon(t) = -\hat{A}^\varepsilon f(t)$$

$$+ \frac{1}{\varepsilon} \int_t^T d\tau [f_z'(X^\varepsilon) K(\tau) E_t^\varepsilon y^\varepsilon(\tau)]_z' \dot{z}^\varepsilon(t)$$

(4.7)
$$+ \frac{1}{\varepsilon} \int_t^T d\tau [f_x(X^\varepsilon) E_t^\varepsilon y^{\varepsilon'}(\tau) M(\tau) z]_z' \dot{z}^\varepsilon(t)$$

$$+ \frac{1}{\varepsilon} \int_t^T d\tau [f_z'(X^\varepsilon) K(\tau) E_t^\varepsilon y^\varepsilon(\tau)]_x \dot{x}^\varepsilon(t)$$

$$+ \frac{1}{\varepsilon} \int_t^T d\tau [f_x(X^\varepsilon) E_t^\varepsilon y^{\varepsilon'}(\tau) M(\tau) z]_x \dot{x}^\varepsilon(\tau).$$

The integral terms on the right of (4.7) are, respectively

$$\frac{1}{\varepsilon^2} \int_t^T d\tau \, E_t^\varepsilon y^{\varepsilon'}(t) K'(t) f_{zz}(X^\varepsilon) K(\tau) y^\varepsilon(\tau)$$

$$\frac{1}{\varepsilon^2} \int_t^T d\tau \, E_t^\varepsilon \{ y^{\varepsilon'}(t) K'(t) f_{xz}(X^\varepsilon) y^{\varepsilon'}(\tau) M(\tau) z^\varepsilon + f_x(X^\varepsilon) y^{\varepsilon'}(t) K'(t) M'(\tau) y^\varepsilon(\tau) \}$$

(4.8)
$$\frac{1}{\varepsilon^2} \int_t^T d\tau [f'_{zx}(X^\varepsilon) K(\tau) E_t y^\varepsilon(\tau)] y^{\varepsilon'}(t) M(t) z^\varepsilon ,$$

$$\frac{1}{\varepsilon^2} \int_t^T d\tau [f_{xx}(X^\varepsilon) E_t^\varepsilon y^{\varepsilon'}(\tau) M(\tau) z] y^{\varepsilon'}(t) M(t) z^\varepsilon .$$

Let $\hat{F}^\varepsilon(X, y^\varepsilon(t), t)/\varepsilon^2$ denote the sum of the terms in (4.8). Define the second test function perturbation $f_2^\varepsilon(t)$ by $f_2^\varepsilon(t) = f_2^\varepsilon(X^\varepsilon(t), t)$, where

$$f_2^\varepsilon(X^\varepsilon, t) = \frac{1}{\varepsilon^2} \int_t^T d\tau [E_t^\varepsilon \hat{F}(X, y^\varepsilon(\tau), \tau) - E \hat{F}(X, y^\varepsilon(\tau), \tau)|_{X=X^\varepsilon}]$$

Also, $f_2^\varepsilon(t) \in \mathcal{D}(\hat{A}^\varepsilon)$ and

$$\hat{A}^\varepsilon f_2^\varepsilon(t) = -\hat{F}(X^\varepsilon, y^\varepsilon(t), t)/\varepsilon^2 + E\hat{F}^\varepsilon(X, y^\varepsilon(t), t)/\varepsilon^2|_{X=X^\varepsilon}$$

$$+ f_2^\varepsilon(X^\varepsilon, y^\varepsilon(t))_x \dot{x}^\varepsilon + f_2^\varepsilon(X^\varepsilon, y^\varepsilon(t))_z' \dot{z}^\varepsilon.$$

Define the (time-dependent) diffusion operator A by

$$Af(X) = \lim_{\epsilon} \frac{1}{\epsilon^2} \hat{L}\hat{F}^\epsilon(X, y^\epsilon(t), t)$$

$$= \lim_{\epsilon \to 0} \int_{t/\epsilon^2}^{T/\epsilon^2} d\tau \, E\Big\{ y'(t/\epsilon^2)K'(t)f_{zz}(X)K(\epsilon^2\tau)y(\tau)$$

$$(4.9) \quad + [y'(\frac{t}{\epsilon^2})K'(t)f_{xz}(X)y'(\tau)M(\epsilon^2\tau)z + y'(\tau)K'(\epsilon^2\tau)f_{xz}(X)y'(t/\epsilon^2)M(t)z]$$

$$+ f_x(X)y'(t/\epsilon^2)K'(t)M'(\epsilon^2\tau)y(\tau)$$

$$+ f_{xx}(X)(y'(\tau)M(\epsilon^2\tau)z)(y'(t/\epsilon^2)M(t)z)\Big\}.$$

The limit of the "f_{zz} component" on the right of (4.9) is trace $\overline{R} \cdot (K'f_{zz}K)/2$. The limit of the "$f_x$ component" is $f_x(X) \sum_{i,j} L_{ij}(t,t)R_{ij}^+$. The limit of the "$f_{xx}$" and "$f_{xz}, f_{zx}$" components are, respectively, $f_{xx}(X)(z'M'\overline{R}Mz)/2$ and trace $\overline{R} \cdot [K'f_{xz}(X)z'M'+Mzf_{xz}'(x)K]/2$. Note that (4.4) is the unique solution $X(\cdot)$ to the martingale problem (of Stroock and Varadhan) for the (time dependent) operator A.

Now, we relate these results to those of [9],[10]. Define the perturbed test function $f^\epsilon(t) = f^\epsilon(t) + f_1^\epsilon(t) + f_2^\epsilon(t)$. In order for $X^\epsilon(\cdot) \to X(\cdot)$ in $D^{r+1}[0,\infty)$ the results of [10] require that (4.10a) to (4.10d) hold for each test function $f(\cdot)$ and each $T < \infty$ and with $X^\epsilon(\cdot)$ varying in an arbitrary bounded set.

$$(4.10a) \qquad \text{p-lim } f_i^\epsilon = 0 \text{ , } i = 1,2$$

$$(4.10b) \qquad \sup_\epsilon P\{\sup_{t \le T}|f_i^\epsilon(t)| \ge N\} \to 0 \text{ as } N \to \infty.$$

$$(4.10c) \qquad f_i^\epsilon \in \mathcal{D}(\hat{A}^\epsilon)$$

$$(4.10d) \qquad \text{p-lim}[\hat{A}^\epsilon f^\epsilon - Af] = 0$$

The method of [9],[10] requires (4.10a,b,c) for $i = 1$ to get tightness of $\{X^\epsilon(\cdot)\}$ in $D^{r+1}[0,\infty)$. All the conditions in (4.10) are then used to identify the operator A and to get $X^\epsilon(\cdot) \Rightarrow X(\cdot)$, the unique solution to the martingale problem for operator A.

Under (A1.a), $f_i^\epsilon(t) = 0(\epsilon^i)[|Y_1^\epsilon(t)|^i + 1]$ and $|\hat{A}^\epsilon f^\epsilon(t) - Af(x^\epsilon(t))| =$
$= 0(\epsilon)[|Y_1^\epsilon(t)|^3 + 1]$. and $P\{\sup_{t \leq T/\epsilon^2} \epsilon|Y_1^\epsilon(t)|^3 \geq \alpha > 0\} \to 0$ for each α. Under
(A1.b), these estimates hold without the $Y_1^\epsilon(t)$ term. Thus (4.10) holds, and the theorem is proved for $m = 1$.

The general case $(m > 1)$ follows from this - simply use the definitions $\mathcal{K} = (K_1,\ldots,K_m)$, $\mathcal{M} = (M_1,\ldots,M_n)$, $Z^\epsilon = (z_1^\epsilon,\ldots,z_m^\epsilon)$ and replace K, M and z^ϵ by $\mathcal{K}, \mathcal{M}$ and Z^ϵ, respectively. Q.E.D.

Weakening the assumptions (A1). Let $y_1^\epsilon(\cdot),\ldots,y_n^\epsilon(\cdot)$ be the zero mean scalar components of the right continuous vector valued $y^\epsilon(\cdot)$ and let $n^\epsilon(\cdot) = y^\epsilon(\cdot)/\epsilon$. Define

(4.11)
$$\hat{f}_{1i}^\epsilon(t) = \int_t^T E_t^\epsilon y_i^\epsilon(s) \, ds/\epsilon$$

$$\hat{f}_{2ij}^\epsilon(t) = \int_t^T d\tau \int_\tau^T [E_t^\epsilon y_i^\epsilon(s)y_j^\epsilon(\tau) - Ey_i^\epsilon(s)y_i^\epsilon(\tau)]ds/\epsilon^2.$$

Let $\sup_{t < \infty, \epsilon > 0} E|y_i^\epsilon(t)|^2 < \infty$. Suppose that $\int_t^T Ey_i^\epsilon(s)y_j^\epsilon(t)ds/\epsilon^2$ is bounded uniformly in $t < T$, $\epsilon > 0$, and for each $t < T$ converges to a limit (not depending on T or t) as $\epsilon \to 0$.

It is apparent from (4.10) that (A1) can be weakened. To verify (4.10a,b) in general, for the f^ϵ constructed in the theorem, we need only that (4.10a,b) hold for each of the $\hat{f}_{1i}^\epsilon$ and $\hat{f}_{2ij}^\epsilon$ of (4.11). The term $\hat{A}^\epsilon \hat{f}^\epsilon$ contains an x-gradient of $f_2^\epsilon(x,t)$ (with x then set equal to $X^\epsilon(t)$) times $\dot{X}^\epsilon(t)$. Thus, to get (4.10d), we need that additionally $p\text{-}\lim \frac{1}{\epsilon}[\hat{f}_{2ij}(t)y_k^\epsilon(t)] = 0$ for all i,j,k. Condition (4.10c) will also hold under these conditions. These conditions are satisfied under (A1).

The perturbed test function method was used in Theorem 1, since reference can readily be made to it. In Chapter 5 of [11] an alternative method (called there the "combined perturbed test function-direct averaging method") is developed. With use of that method for proving weak convergence, and for the case $y^\epsilon(t) = y(t/\epsilon^2)$, (A1) can be replaced by (4.12)

(4.12a) $\{|y(t)|^2, \sup_{\Delta<1}|\int_{t+\Delta}^T E_t y(u)du|^2, t < \infty, T < \infty\}$ uniformly integrable

(4.12b) $\int_t^T E_t y(u)du$ continuous in probability, uniformly in $t \leq T$, for each T.

(4.12c) $\int_s^\tau E\, y(u)y(s)'du$ converges as $\tau-s \to \infty$

(4.12d) $E|\int_s^\tau du[E_t y(u)y'(s)-Ey(u)y'(s)]| \to 0$ as $\tau-s\to\infty$, $s-t\to\infty$.

These conditions (which are just (5.8.20) to (5.8.23) in [11]) are rather unrestrictive, as are the requirements associated with (4.11). In fact (4.12) is implied by (A1).

Theorem 2. Assume the model (4.2), (4.3) and contition (A1). The process defined by $\int_0^\cdot y^\epsilon(u)du/\epsilon$ converges weakly to $\sqrt{\bar{R}}\, w(\cdot)$. Let M_i and K_i be in $L_2[0,T]$ for each $T < \infty$, but not necessarily continuous. Then $z_i^\epsilon \to z_i$ of (4.5) and the finite dimensional distributions of $x^\epsilon(\cdot)$ converge to those of (4.5). If the $M_i(\cdot)$ are bounded on each $[0,T]$, then the weak convergence of Theorem 1 continues to hold.

Proof. The first assertion is from Theorem 1. It is sufficient to work with $m = 1$. The value of the constant k (depending only on T) might change from one usage to another. For notational simplicity, we consider only the scalar and Gaussian case (which is slightly harder than the ϕ-mixing case, since $y(\cdot)$ is unbounded). In the ϕ-mixing case, we simply replace (4.14) by in-

equalities derived from (2.3).

Part 1. We have

$$(4.13) \quad E|z^{\epsilon}(t+\delta)-z^{\epsilon}(t)|^4 = \frac{1}{\epsilon^4} \int_t^{t+\delta} dt_1 dt_2 dt_3 dt_4 K(t_1) \ldots K(t_4) Ey^{\epsilon}(t_1) \ldots y^{\epsilon}(t_4).$$

Define $R^{\epsilon}(t) = (|R(t/\epsilon^2)| + |R(-t/\epsilon^2)|)/\epsilon^2$, for $|t| < \infty$. Then

$$(4.14) \quad |Ey^{\epsilon}(t_1) \ldots y^{\epsilon}(t_4)| \leq \epsilon^4 [R^{\epsilon}(t_2-t_1)R^{\epsilon}(t_4-t_3)$$

$$+ R^{\epsilon}(t_3-t_1)R^{\epsilon}(t_2-t_4) + R^{\epsilon}(t_4-t_1)R^{\epsilon}(t_2-t_3)].$$

Define $\int_t^{t+\delta} |K(s)| R^{\epsilon}(\tau-s) ds = K_{\epsilon}(\tau)$. Then

$$
\begin{aligned}
\int_t^{t+\delta} K_{\epsilon}^2(\tau) d\tau &= \int_t^{t+\delta} d\tau (\int_t^{t+\delta} |K(s)| R^{\epsilon}(\tau-s) ds)^2 \\
(4.15) \quad &\leq \int_t^{t+\delta} d\tau (\int_t^{t+\delta} K^2(s) R^{\epsilon}(\tau-s) ds) \int_0^{\infty} R^{\epsilon}(s) ds \\
&\leq k(\int_0^{\infty} R^{\epsilon}(\tau) d\tau)^2 \int_t^{t+\delta} K^2(s) ds.
\end{aligned}
$$

The expression (4.13) is upper bounded by

$$k(\int_t^{t+\delta} |K(\tau)| K_{\epsilon}(\tau) d\tau)^2$$

Then, by the Schwarz inequality and (4.15)

$$(4.16a) \quad (4.13) \leq k(\int_t^{t+\delta} K^2(s) ds)^2.$$

But ([3], Theorem 12.3), (4.16a) implies that $\{z^{\epsilon}(\cdot), \epsilon > 0\}$ is tight in $C^r[0,\infty)$ for each K. Also for each $\delta > 0$ and $T < \infty$, (4.16a) implies that

$$(4.16b) \quad \sup_{\epsilon>0} P\{\sup_{t \leq T} |z^{\epsilon}(t)| > \delta > 0\} \to 0 \quad \text{as} \quad ||K|| \to 0.$$

Part 2. Let us next evaluate $E(x^{\epsilon}(t))^2$:

$$E(x^\epsilon(t))^2 = E[\int_0^t d\tau \, y^\epsilon(\tau)M(\tau)\int_0^\tau ds \, y^\epsilon(s)K(s)]^2/\epsilon^4$$

$$(4.17) \qquad \le \int_0^t ds_1 ds_2 ds_3 ds_4 |M(s_1)||M(s_2)||K(s_3)||K(s_4)| \cdot$$

$$\cdot [R^\epsilon(s_2-s_1)R^\epsilon(s_4-s_3)+ R^\epsilon(s_3-s_1)R^\epsilon(s_2-s_4)+R^\epsilon(s_4-s_1)R^\epsilon(s_3-s_2)]$$

Let us evaluate the part of the integral on the right of (4.17) containing $R^\epsilon(s_3-s_1)R^\epsilon(s_2-s_4)$ only. Write

$$M_\epsilon(\tau) = \int_0^t |M(s)|R^\epsilon(\tau-s)ds.$$

Then $\int_0^t M_\epsilon^2(\tau)d\tau \le k \int_0^t |M(\tau)|^2 d\tau$, and similarly for $K(\cdot)$. Using this, we obtain that the term is bounded above by

$$(4.18) \qquad \int_0^t M_\epsilon(s_3)|K(s_3)|ds_3 \int_0^t K_\epsilon(s_2)|M(s_2)|ds_2$$

$$\le k||M||^2||K||^2.$$

Via a similar analysis, we obtain that the other terms of (4.17) are also bounded by the right side of (4.18).

Part 3. Let $\tilde{M}_n$ be continuous for each n, and define $\delta\tilde{M}_n = M - \tilde{M}_n$, where $||\delta M_n|| \to 0$. Define x_n^ϵ by $x_n^\epsilon(0) = 0$ and

$$(4.19) \qquad \dot{x}_n^\epsilon = \tilde{M}_n z^\epsilon y^\epsilon/\epsilon.$$

Then

$$(4.20) \qquad \dot{x}^\epsilon = Mz^\epsilon y^\epsilon/\epsilon = \tilde{M}_n^\epsilon z^\epsilon y^\epsilon/\epsilon + \delta M_n z^\epsilon y^\epsilon/\epsilon,$$

Using the tightness of $\{z^\epsilon(\cdot)\}$, an argument like that in Theorem 1 can be used to obtain that $\{z^\epsilon(\cdot), x_n^\epsilon(\cdot), \epsilon > 0\}$ is tight in $D^{r+1}[0,\infty)$ for each n and that the weak limit satisfies (4.4) with M replaced by $\tilde{M}_n$.

Now, applying the estimate in Part 2 to the case where δM_n replaces M, we get

$$\sup_{\varepsilon>0} E\left(\int_0^t ds\, \delta M_n(s) z^\varepsilon(s) y^\varepsilon(s)/\varepsilon\right)^2 = \sup_{\varepsilon>0} E\left|x^\varepsilon(t) - x_n^\varepsilon(t)\right|^2 \to 0 \quad \text{as } n \to \infty.$$

The second assertion of the theorem follows from this and the above cited weak convergences.

$\underline{\text{Part 4.}}$ Let M_i be bounded on $[0,T]$. Again, it is sufficient to work with $m = 1$. Write $M = M_1$ and define $\tilde{M}_n, \delta M_n$ as in Part 3. Define $\hat{x}_n^\varepsilon(t) = x^\varepsilon(t) - x_n^\varepsilon(t) = \int_0^t ds\, \delta M_n(s) z^\varepsilon(s) y^\varepsilon(s)/\varepsilon$. To prove the last assertion of the theorem, we need only prove for each $T < \infty$ and $\delta > 0$, that

(4.21)
$$\lim_{\varepsilon>0} P\{\sup_{t\le T} |\hat{x}_n^\varepsilon(t)| > \delta > 0\} \to 0$$

as $||\delta M_n|| \to 0$, where we can assume that $\{\tilde{M}_n(\cdot)\}$ and $M(\cdot)$ have a common finite upper bound. Since $\{z^\varepsilon(\cdot)\}$ is tight in $D^r[0,\infty)$ by Part 1, by changing the value of $z^\varepsilon(\cdot)$ (for each $\varepsilon > 0$) on a set of arbitrarily small probability we can suppose that there is an $N < \infty$ such that $\sup_{t\le T} |z^\varepsilon(t)| < N$. If (4.21) holds with this bound, where N is arbitrary, then it holds as stated. Henceforth, we let $|z^\varepsilon(t)|$ be bounded by N and absorb N into the constants k, where appropriate.

A "perturbed Liapunov function" method will be used. Define $W(x) = x^2$. Then $\hat{A}^\varepsilon W(\hat{x}_n^\varepsilon(t)) = 2\hat{x}_n^\varepsilon(t) z^\varepsilon(t) \delta M_n(t) y^\varepsilon(t)/\varepsilon$. Recall that $R(t) = H_1 R_1(t)$ and

define $R^\epsilon(\cdot)$ as in Part 1. Define the perturbation $W_1^{\epsilon,n}(\cdot)$ and $W^{\epsilon,n}(\cdot)$ by (write $\hat{x} = \hat{x}_n^\epsilon(t)$ and $z = z^\epsilon(t)$, where convenient) by $W^{\epsilon,n}(t) = W(\hat{x}) + + W_1^{\epsilon,n}(t)$, where

$$W_1^{\epsilon,n}(t) = \int_t^T 2\hat{x}z\,\delta M_n(s)E_t y^\epsilon(s)\,ds/\epsilon$$

$$= \frac{1}{\epsilon}\int_t^T 2\hat{x}z\,\delta M_n(s)H_1R_1(\frac{s-t}{\epsilon^2})Y_1^\epsilon(t)\,ds.$$

Define $\delta M_n^\epsilon(t) = \int_t^T |\delta M_n(s)|R^\epsilon(s-t)\,ds$. Since $\delta M_n^\epsilon(\cdot)$ is bounded on $[0,T]$, uniformly in ϵ and n, we have (change variables $s/\epsilon^2 \to s$, and recall that we are assuming that $z^\epsilon(t)$ is bounded)

$$|W_1^{\epsilon,n}(t)| \leq k\,\epsilon\,|\hat{x}|\,\delta M_n^\epsilon(t)\,|Y_1^\epsilon(t)|$$

(4.22)

$$\leq k\epsilon|\hat{x}|^2 + k\epsilon|Y_1^\epsilon(t)|^2 \,\delta M_n^\epsilon(t)$$

Define $F(\hat{x},z,s,t) = 2\hat{x}z\,\delta M_n(s)E_t^\epsilon y^\epsilon(s)/\epsilon$. Then (for almost all t)

$$\hat{A}^\epsilon W_1^{\epsilon,n}(t) = -2\hat{x}z\,\delta M_n(t)y^\epsilon(t)/\epsilon + \int_t^T F_x(\hat{x},z,s,t)\,ds\,\dot{\hat{x}}_n^\epsilon(t)$$

$$+ \int_t^T F_z(\hat{x},z,s,t)\,ds\,\dot{z}^\epsilon(t).$$

Thus (to obtain the second line, change variables $s/\epsilon^2 \to s$ and use (A1))

$$\hat{A}^\epsilon W^{\epsilon,n}(t) = \frac{2}{\epsilon^2}\int_t^T ds\,\delta M_n(s)E_t^\epsilon y^\epsilon(s)\,[\hat{x}K(t)y^\epsilon(t) + z^2\delta M_n(t)y^\epsilon(t)],$$

(4.23) $\quad E|\hat{A}^\epsilon W^{\epsilon,n}(t)| \leq k\,\delta M_n^\epsilon(t)\,|K(t)|E^{\frac{1}{2}}|\hat{x}|^2 + k\delta M_n^\epsilon(t)\,|\delta M_n(t)|$

$$\leq k|\delta M_n^\epsilon(t)|^2\,|K(t)|^2 + kE|\hat{x}|^2 + k\,\delta M_n^\epsilon(t)\,|\delta M_n(t)|$$

The expression $F^{\epsilon,n}(\cdot)$ in (4.24) is a martingale, with $F^\epsilon(0) = 0$ (see Section 3).

(4.24) $$F^{\epsilon,n}(t) = W^{\epsilon,n}(t) - \int_0^t \hat{A}^\epsilon W^{\epsilon,n}(s)\,ds.$$

Taking expectations in (4.24) and using the bounds (4.22), (4.23) and
$\int_0^T \delta M_n^\epsilon(t) |\delta M_n(t)| dt \le ||\delta M_n||^2$ yields (for $t \le T$)

$$(4.25) \qquad E|\hat{x}_n^\epsilon(t)|^2 \le k\int_0^t E|\hat{x}_n^\epsilon(s)|^2 ds + k||\delta M_n||^2 + k\int_0^T (\delta M_n^\epsilon(t))^2 K^2(t) dt + \epsilon k$$

Note that $\sup_{\epsilon>0} \sup_{t\le T} \delta M_n^\epsilon(t)$ is bounded and that $\delta M_n^\epsilon(\cdot) \to$ zero function in measure (uniformly in $\epsilon \to 0$) as $n \to \infty$. This and (14.25) imply

$$(4.26) \qquad \overline{\lim_{\epsilon>0}} \sup_{t\le T} E|\hat{x}_n^\epsilon(t)| \le \delta_n = k[||\delta M_n||^2 + \int_0^T (\delta M_n^\epsilon(t))^2 K^2(t) dt] \xrightarrow{n} 0.$$

Since $F^{\epsilon,n}(\cdot)$ is a martingale,

$$(4.27) \qquad P\{\sup_{t\le T} |F^{\epsilon,n}(t)| \ge a > 0\} \le E|F^{\epsilon,n}(T)|/a.$$

We have for each $b > 0$

$$(4.28) \quad \overline{\lim_{\epsilon>0}} P\{\sup_{t\le T} |\int_0^t \hat{A}^\epsilon W^{\epsilon,n}(s) ds| \ge b > 0\} \le E\int_0^T |\hat{A}^\epsilon W^{\epsilon,n}(s)| ds/b \le k\delta_n/b,$$

$$\overline{\lim_{\epsilon>0}} P\{\sup_{t\le T/\epsilon^2} \epsilon |Y_1(t)|^2 \ge b > 0\} = 0.$$

The bound (4.22) and estimates (4.28) substituted into (4.27) yield (4.21). Q.E.D.

__Theorem 3.__ __Assume__ (A1). __For the general form__ (1.1a) __where__ $m = \infty$, __the finite dimensional distributions of__ $x^\epsilon(\cdot)$ __converge to those of__ (4.6), __where__ $L(t,t) = \sum_i \lambda_i M_i(t) K_i(t)$ __and__ $\int_0^\cdot y^\epsilon(s) ds/\epsilon \Rightarrow \sqrt{R} w(\cdot)$. __Assume__ (A1.b), __and let the__ $\{M_i, K_i\}$ __be uniformly bounded on each finite time interval.__ __Then__ (4.6) __is the weak limit of__ $\{x^\epsilon(\cdot)\}$ __in__ $D[0,\infty)$.

__Proof.__ __Part 1.__ In view of Theorem 2, for the first assertion it is sufficient ot show that for each $t < \infty$,

$$(4.29) \qquad E|\sum_n^\infty \lambda_i \int_0^t y^\epsilon(\tau)'M_i(\tau) d\tau \int_0^\tau K_i(s) y^\epsilon(s) ds/\epsilon^2|^2 \to 0$$

as $n \to \infty$, uniformly in $\varepsilon \to 0$. We only outline the (straightforward) calculation in the scalar-Gaussian case. In this case,

$$(4.29) \leq \sum_n^{\infty} \sum |\lambda_i \lambda_j| \int_0^t ds_1 ds_2 ds_3 ds_4 |M_i(s_1) K_i(s_2) M_j(s_3) K_j(s_4)| \cdot$$

$$\cdot [R^{\varepsilon}(s_2 - s_1) R^{\varepsilon}(s_4 - s_3) + R^{\varepsilon}(s_3 - s_1) R^{\varepsilon}(s_2 - s_4) + R^{\varepsilon}(s_4 - s_1) R^{\varepsilon}(s_3 - s_2)].$$

The evaluation now proceeds as in Theorem 2, part 2, to yield the bound

$$(4.29) \leq \sum_n^{\infty} \sum \lambda_i \lambda_j ||M_i|| \; ||M_j|| \; ||K_i|| \; ||K_j||$$

$$\leq (\sum_n^{\infty} \lambda_i ||M_i|| \; ||K_i||)^2 \underset{n}{\to} 0.$$

Part 2. Note that $\sum_n^{\infty} |\lambda_i x_i^{\varepsilon}(t)| \leq \sum_n^{\infty} |\lambda_i| + \sum_n^{\infty} |\lambda_i| \; |x_i^{\varepsilon}(t)|^2$. Thus for the second assertion, in view of Theorems 1 and 2, it is sufficient to assume $\lambda_i \geq 0$ and to show that for each $T < \infty$ and $\delta > 0$

$$(4.30) \qquad \sup_{\varepsilon > 0} P\{\sup_{t \leq T} \sum_n^{\infty} \lambda_i |x_i^{\varepsilon}(t)|^2 \geq \delta > 0\} \underset{n}{\to} 0.$$

To simplify the notation, let all the K_i, M_i, x_i^{ε}, z_i^{ε} be scalar valued. The bounds on K_i, M_i will be absorbed into the k and $O(\varepsilon)$ below. A perturbed Liapunov function argument of the type used in Theorem 2, Part 4, will be employed. First we show

$$(4.31) \qquad \begin{aligned} &\sup_{\varepsilon > 0} P\{\sup_{t \leq T} \sum_n^{\infty} \lambda_i |z_i^{\varepsilon}(t)|^2 \geq \delta > 0\} \underset{n}{\to} 0 \\ &\sup_{\varepsilon > 0} \sup_{t \leq T} E \sum_n^{\infty} \lambda_i |z_i^{\varepsilon}(t)|^2 \underset{n}{\to} 0. \end{aligned}$$

Define $V_n(z) = \sum_n^{\infty} \lambda_i |z_i|^2$ and $V_n^{\varepsilon}(t) = V_n(z^{\varepsilon}(t))$. Then $\hat{A}^{\varepsilon} V_n^{\varepsilon}(t) = 2 \sum \lambda_i z_i^{\varepsilon}(t) K_i(t) y^{\varepsilon}(t)/\varepsilon$. Define the perturbation $\delta V_n^{\delta}(t)$ by

$$\delta V_n^{\varepsilon}(t) = \frac{1}{\varepsilon} \int_t^T \sum_n^{\infty} \lambda_i z_i^{\varepsilon}(t) K_i(s) E_t^{\varepsilon} y^{\varepsilon}(s) ds.$$

Define $\hat{V}_n^\epsilon(t) = V_n^\epsilon(t) + \delta V_n^\epsilon(t)$. Under (A1.b) we have

$$|\delta V_n^\epsilon(t)| = 0(\epsilon) \sum_n^\infty \lambda_i |z_i^\epsilon(t)|$$

$$\leq 0(\epsilon) \sum_n^\infty \lambda_i [1 + |z_i^\epsilon(t)|^2|.$$

$$\hat{A}^\epsilon \hat{V}_n^\epsilon(t) = \frac{2}{\epsilon^2} \int_t^T \sum_n^\infty \lambda_i K_i(s) K_i(t) y^\epsilon(t) E_t^\epsilon y^\epsilon(s) ds$$

$$= 0(1) \sum_n^\infty \lambda_i$$

Also $\hat{V}_n^\epsilon(t) - \int_0^t \hat{A}^\epsilon \hat{V}_n^\epsilon(s) ds$ is a martingale. Now, following the procedures in Theorem 2, Part 4, we obtain that for each $\delta > 0$

$$\sup_{t \leq T} E \sum_n^\infty \lambda_i |z_i^\epsilon(t)|^2 \leq k \sum_n^\infty \lambda_i$$

(4.32)

$$\sup_{\epsilon > 0} P\{ \sup_{t \leq T} \sum_n^\infty \lambda_i |z_i^\epsilon(t)|^2 \geq \delta > 0\} \xrightarrow{n} 0.$$

Next, the procedure is repeated for $W_n(x) = \sum_n^\infty \lambda_i |x_i^\epsilon(t)|^2$. We have $\hat{A}^\epsilon W_n(x^\epsilon(t)) = 2 \sum_n^\infty \lambda_i x_i^\epsilon(t) M_i(t) z_i^\epsilon(t) y^\epsilon(t)/\epsilon$. Define the perturbation

$$\delta W_n^\epsilon(t) = \frac{2}{\epsilon} \int_t^T \sum_n^\infty \lambda_i z_i^\epsilon(t) M_i(s) x_i^\epsilon(t) E_t^\epsilon y^\epsilon(s) ds$$

and set $\hat{W}_n^\epsilon(t) = W_n(x^\epsilon(t)) + \delta W_n^\epsilon(t)$. Now (A1.b) implies that

$$|\delta W_n^\epsilon(t)| = 0(\epsilon) \sum_n^\infty \lambda_i |z_i^\epsilon(t)| |y_i^\epsilon(t)|$$

$$\leq 0(\epsilon) \sum_n^\infty \lambda_i [|z_i^\epsilon(t)|^2 + |x_i^\epsilon(t)|^2].$$

Analogously to the procedure of Theorem 2,

$$\hat{A}^\epsilon \hat{W}_n^\epsilon(t) = \frac{2}{\epsilon^2} \int_t^T ds \sum_n^\infty \lambda_i M_i(s) E_t^\epsilon y^\epsilon(s) [(z_i^\epsilon(t))^2 M_i(t) + x_i^\epsilon(t) K_i(t)] y^\epsilon(t)$$

$$|\hat{A}^\epsilon \hat{W}_n^\epsilon(t)| \leq k \sum_n^\infty \lambda_i [(z_i^\epsilon(t))^2 + |x_i^\epsilon(t)|].$$

As for the above case for $V_n(z^\epsilon)$, these estimates and the martingale property of $\hat{W}_n^\epsilon(t) - \int_0^t \hat{A}^\epsilon \hat{W}_n^\epsilon(s)\,ds$ yield

(4.33)
$$\sup_{t \le T} E \sum_n^\infty \lambda_i |x_i^\epsilon(t)|^2 \le \sup_{t \le T} k\, E \sum_n^\infty \lambda_i |z_i(t)|^2 \le k \sum_n^\infty \lambda_i.$$

This and the above martingale property yield the derived estimate (4.30), similarly to what was done in Theorem 2, Part 4. Q.E.D.

5. An Application to the Approximation of a Likelihood Ratio.

We work with the system

$$dx = Ax\,dt + Cdw_1$$
(5.1)
$$dy = Hx\,dt + dw_2$$

where $w_1(\cdot)$ and $w_2(\cdot)$ are standard and mutually independent Wiener processes. The $x(\cdot)$ and $y(\cdot)$ here are <u>not</u> the same as those in the previous sections. Let $\hat{x}(t) = E[x(t)|y(u), u \le t]$. Then on each $[0,T]$,(the likelihood functional) Radon-Nikodyn derivative of the measure of (5.1) to that of (5.2) is the $F(\cdot)$ of (5.3)

$$dx = Ax\,dt + Cdw_1$$
(5.2)
$$dy = dw_2.$$

$$F(t) = \exp f(t),$$
(5.3)
$$f(t) = -\frac{1}{2} \int_0^t |Hx(\tau)|^2 d\tau + \int_0^t dy'(\tau)Hx(\tau)\,d\tau.$$

Also

$$d\hat{x} = A\hat{x}\,dt + G(dy - H\hat{x}\,dt)$$
$$= (A - GH)\hat{x}\,dt + Gdy,$$

where $G(\cdot)$ is obtainable from the solution to the Riccati equation. We can write

$$(5.4) \qquad \hat{x}(t) = \phi(t,0)\hat{x}(0) + \int_0^t \phi(t,s)G(s)dy(s) \equiv \int_0^t U(t)V(s)Hx(s)ds$$

$$+ \int_0^t U(t)V(s)dw_2(s)$$

$$+ \phi(t,0)\hat{x}(0),$$

where $\phi(\cdot,\cdot)$ is the fundamental matrix of $\dot{v} = (A - GH)v$, and $U(\cdot)$ and $V(\cdot)$ are continuous. The second integral of (5.3) equals

$$(5.5) \qquad \int_0^t d\tau[Hx(\tau)]' H \int_0^\tau U(\tau)V(s)[Hx(s)ds + dw_2(s)]$$

$$+ \int_0^t dw_2(\tau)' H \int_0^\tau U(\tau)V(s)Hx(s)ds$$

$$+ \int_0^t dw_2'(\tau) H \int_0^\tau U(\tau)V(s)dw_2(s) + \int_0^t [dw_2(\tau)+Hx(\tau)\iota\tau]'HU(\tau)V(0)\hat{x}(0)$$

Consider the problem where "wide-band" noise $\psi^\varepsilon(\cdot)/\varepsilon$ replaces the ideal infinite band-width white observation noise $\dot{w}_2(\cdot)$, and where $\psi^\varepsilon(\cdot) = \psi(t/\varepsilon^2)$ and $\psi(\cdot)$ satisfies (A1), and is independent of $w_1(\cdot)$. Under (A1), $\int \psi^\varepsilon(s)ds/\varepsilon \to w_2(\cdot)$, a Wiener process with covariance $\bar{R}t$. W.l.o.g., assume $\bar{R}=I$, since the components of the observation noise in (5.1) are mutually independent. In the general case, simply replace all dw_2 by $\sqrt{\bar{R}}\,dw_2$. As in [1], an important approximation and limit problem arises, since we must use the "physical" wide-band noise $[\psi^\varepsilon(s)/\varepsilon]ds$ in lieu of the ideal $dw_2(s)$ in (5.5). Replace $dw_2(s)$ and $dw_2(\tau)$ by $[\psi^\varepsilon(s)/\varepsilon]ds$ and $[\psi^\varepsilon(\tau)/\varepsilon]d\tau$, respectively, in (5.1)-(5.5). Let $\hat{x}^\varepsilon(\cdot)$ denote the value of (5.4) obtained with this replacement. Then both $\{\hat{x}^\varepsilon(\cdot)\}$ and the first two integrals of (5.5) (with the ψ-replacement) are tight and converge weakly to (5.4) and those integrals.

Also, the limit $w_2(\cdot)$ is independent of $w_1(\cdot)$. The only problem is the 3rd integral of (5.5), which we rewrite (with the ψ^ε replacement) as

$$(5.6) \qquad \frac{1}{\varepsilon^2} \int_0^t d\tau \, \psi^{\varepsilon'}(\tau) H \int_0^\tau U(\tau)V(s)\psi^\varepsilon(s)\,ds = u^\varepsilon(t).$$

Write $dy^\varepsilon(t) = [H\hat{x}^\varepsilon(t) + \psi^\varepsilon(t)/\varepsilon]dt$ and

$$\dot{z}^\varepsilon = V\psi^\varepsilon/\varepsilon \ , \ z^\varepsilon(0) = 0$$

$$(5.7)$$

$$\dot{u}^\varepsilon = \psi^{\varepsilon'}HUz^\varepsilon/\varepsilon \ , \ u^\varepsilon(0) = 0$$

Theorem 1 can be immediately applied to yield that $\{z^\varepsilon(\cdot), u^\varepsilon(\cdot)\}$ converges weakly to $(z(\cdot), u(\cdot))$, where

$$dz = Vdw_2, \ R_{ij}^+ = \int_0^\infty E\psi_i(s)\psi_j(0)\,ds$$

$$(5.8)$$

$$du = \frac{1}{2} \sum_{i,j} (HU(t)V(t))_{ij} R_{ij}^+ dt + z'U'H'dw_2$$

In general, R_{ij}^+ need not be zero for $i \neq j$. In the special case where the $\{\psi_i(\cdot)\}$ are mutually independent and $\overline{R} = I$, we get $R_{ij}^+ = \delta_{ij}/2$ and

$$(5.9) \qquad u(t) = \frac{1}{2}\int_0^t \text{trace } HU(s)V(s)\,ds + \int_0^t dw_2(\tau)HU(\tau)\int_0^\tau V(s)\,dw_2(s)$$

Define

$$f^\varepsilon(t) = -\frac{1}{2}\int_0^t |H\hat{x}^\varepsilon(s)|^2 ds + \int_0^t [Hx(s)+\psi^\varepsilon(s)/\varepsilon]H\hat{x}^\varepsilon(s)\,ds$$

$$- \frac{1}{2}\int_0^t \sum_{i,j}(HU(s)V(s))_{ij}R_{ij}^+\,ds \ ,$$

$$F^\varepsilon(t) = \exp f^\varepsilon(t).$$

By our results, the system $\{x(\cdot), \hat{x}^\varepsilon(\cdot), y^\varepsilon(\cdot), F^\varepsilon(\cdot)\}$ converges weakly to the system $\{x(\cdot), \hat{x}(\cdot), y(\cdot), F(\cdot)\}$ as $\varepsilon \to 0$.

Remarks. $F^\varepsilon(\cdot)$ is <u>not</u> a likelihood functional. But the simultaneous weak convergence of the system and functional just cited implies that it can be used in statistical tests (for small $\varepsilon > 0$); for example in a hypothesis test for testing whether H takes the value H_0 or 0 (signal or no signal, respectively). This is because for each $a > 0$

$$P\{F^\varepsilon(t) > a | H = H_0\} \lessgtr P\{F(t) > a | H = H_0\}.$$

Then, the use of $F^\varepsilon(\cdot)$ yields an "approximation" to the test which we would have under the assumption that $\dot{w}_2(\cdot)$ were the actual observation noise. The weak convergence is important if the test is to be meaningful - the convergence of finite dimensional distributions (as obtained in [1]) is not actually enough, because it is essential that the "system" $\hat{x}^\varepsilon(\cdot)$ approximate $\hat{x}(\cdot)$ also, for otherwise we do not have an approximation to a likelihood functional for a particular system.

Owing to the nature of weak convergence, the distributions of the passage times of $F^\varepsilon(\cdot)$ through any given level also converges. Thus, the results can also be used for a <u>sequential test</u>. A weak convergence type of result is essential for this. It could not be done if only finite dimensional distributions converged. We end by remarking again that the procedure works with many other types of noise processes.

6. Approximation of Multiple Integrals (order > 2)

In order to simplify what would otherwise be a very awkward notation, let $y(\cdot)$ and $L(\cdot)$ be scalar valued. We seek the (weak) limit of

$$(6.1) \quad x^\varepsilon(t) = \int_0^t ds_m \int_0^{s_m} ds_{m-1} \cdots \int_0^{s_2} ds_1 L(s_1, \ldots, s_m) y^\varepsilon(s_1) \ldots y^\varepsilon(s_m) / \varepsilon^m,$$

where $y^\epsilon(t) = y(t/\epsilon^2)$. Retain (Al), although the condition can be consider-ably weakened. Also the $\prod_i y^\epsilon(s_i)$ can be replaced by $\prod_i y_i^\epsilon(s_i)$ if the set $\{y_1(\cdot),\ldots,y_m(\cdot)\}$ satisfies (Al). Only an informal outline will be given. We first let $L(s_1,\ldots,s_m) = \prod_1^m L_i(s_i)$, where each $L_i(\cdot)$ is continuous.

Define, recursively, $x_0^\epsilon(t) \equiv 1$ and $(x_i^\epsilon(0) = 0$ for $i > 0)$

$$\dot{x}_1^\epsilon = L_1^\epsilon y^\epsilon/\epsilon = L_1 x_0^\epsilon y^\epsilon/\epsilon$$

(6.2)
$$\dot{x}_2^\epsilon = L_2 x_1^\epsilon y^\epsilon/\epsilon$$
$$\vdots$$
$$\dot{x}_m^\epsilon = L_m x_{m-1}^\epsilon y^\epsilon/\epsilon.$$

Then $x_m^\epsilon(\cdot) = x^\epsilon(\cdot)$. Either the method of Theorem 1 or the (preferable "di-rect averaging") method mentioned after Theorem 1 can be used to obtain the correct limit.

<u>Theorem 4</u>. <u>Under</u> (Al), $\{x_i^\epsilon(\cdot), i \leq m\} \Rightarrow (x_i(\cdot), i \leq m)$, <u>where</u> $(\bar{R} = \int_{-\infty}^{\infty} R(u)du,$ <u>and</u> $w(\cdot)$ <u>is a standard Wiener process)</u>

(6.3)
$$\begin{pmatrix} dx_1 \\ dx_2 \\ \cdot \\ \vdots \\ \cdot \\ dx_m \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{R}L_2 L_1 x_0 dt/2 \\ \cdot \\ \\ \bar{R}L_m L_{m-1} x_{m-2} dt/2 \end{pmatrix} + \sqrt{\bar{R}} \begin{pmatrix} L_1 \\ L_2 x_1 \\ \cdot \\ \vdots \\ \cdot \\ L_m x_{m-1} \end{pmatrix} dw$$

<u>Remark</u>. The "correction term" is more complicated here. It can be obtained directly by solving (6.3) for $x_m(\cdot)$. The correction term is the sum of all the terms in $x_m(t)$ except for the m-dimensional itô integral

$$\bar{R}^{m/2} \int_0^t \int_0^{s_m} \cdots \int_0^{s_2} \prod_1^m L_i(s_i) dw(s_i).$$

The proof is essentially the same as that of Theorem 1 and we only make a few remarks.

To use the method of Theorem 1, choose a test function $f(x_1,\ldots,x_m)$ whose second partial derivatives are continuous and which has compact support. Then (write $x_i^\epsilon = x_i^\epsilon(t)$, where convenient)

$$\hat{A}^\epsilon f(x_1,\ldots,x_m) = \sum_i^m f_{x_i}(x_1^\epsilon,\ldots,x_m^\epsilon)\dot{x}_i^\epsilon = \sum_i^m f_{x_i}(x_1,\ldots,x_m)L_i x_{i-1}^\epsilon y^\epsilon/\epsilon$$

Analogously to the method of Theorem 1, define the perturbations $f_1(t)$ and $f^\epsilon(t) = f(x_1^\epsilon,\ldots,x_m^\epsilon) + f_1^\epsilon(t)$ by

$$f_1^\epsilon(t) = \frac{1}{\epsilon}\int_t^T \sum f_{x_i}(x_1^\epsilon,\ldots,x_m^\epsilon)L_i(s)x_{i-1}^\epsilon E_t^\epsilon y^\epsilon(s)ds.$$

Then (analogously to the case in Theorem 1)

$$\hat{A}^\epsilon f^\epsilon(t) = \frac{1}{\epsilon}\sum_{i,j=1}^m \int_t^T f_{x_i x_j}(x_1^\epsilon,\ldots,x_m^\epsilon)L_i(s)x_{i-1}^\epsilon E_t^\epsilon y^\epsilon(s)ds\ \dot{x}_j^\epsilon$$

(6.4)

$$+ \frac{1}{\epsilon}\sum_{i=1}^m \int_t^T f_{x_i}(x_1^\epsilon,\ldots,x_m^\epsilon)L_i(s)E_t^\epsilon y^\epsilon(s)ds\ \dot{x}_{i-1}^\epsilon$$

which equals (change variables $s/\epsilon^2 \to s$ and substitute for $\dot{x}_j^\epsilon$)

$$\sum_{i,j=1}^m \int_{t/\epsilon^2}^{T/\epsilon^2} f_{x_i x_j}(x_1^\epsilon,\ldots,x_m^\epsilon)x_{i-1}^\epsilon L_i(\epsilon^2 s)E_t^\epsilon y(s)ds[L_j(t)y^\epsilon(t)x_{j-1}^\epsilon]$$

(6.5)

$$+ \sum_{i=2}^m \int_{t/\epsilon^2}^{T/\epsilon^2} f_{x_i}(x_1^\epsilon,\ldots,x_m^\epsilon)L_i(\epsilon^2 s)E_t^\epsilon y(s)ds[L_{i-1}(t)x_{i-2}^\epsilon y^\epsilon(t)].$$

To obtain the operator A of the limit process, take expectations in (6.5) and let $\epsilon \to 0$ to obtain (the time dependent operator).

$$(6.6)\quad Af(x_1,\ldots,x_m) = \frac{1}{2}\sum_{i,j=1}^m \bar{R}\, f_{x_i x_j}(x_1,\ldots,x_m)L_i(t)L_j(t)x_{i-1}x_{j-1}$$

$$+ \frac{\bar{R}}{2}\sum_{i=2}^m f_{x_i}(x_1,\ldots,x_m)L_i(t)L_{i-1}(t)x_{i-2}$$

This is the operator of (6.3). The component with the f_{x_i} yields the "correction term".

Now, let

(6.7) $\qquad L(s_1,\ldots,s_m) = \sum_{j=1}^{n} \prod_{i=1}^{m} L_i^j(s_i) \equiv \sum_{j=1}^{n} L^j(s_1,\ldots,s_m)$

where the $L_i^j(\cdot)$ are continuous. The procedure is, again, an extension of that of Theorem 1. Each integral (for $j=1,\ldots,n$)

$$x^{\varepsilon,j}(t) = \int_0^t ds_m \cdot \int_0^{s_m} ds_{m-1} \cdots \int_0^{s_2} L^j(s_1,\ldots,s_m) y^\varepsilon(s_1)\ldots y^\varepsilon(s_m)/\varepsilon^m$$

is "state variabilized" and treated separately, except that the limit Wiener process $w(\cdot)$ does not depend on j here-since the $y^\varepsilon(\cdot)$ don't depend on j here. Thus $x^\varepsilon(t) \Rightarrow x(\cdot) = \sum_{j=1}^{n} \bar{x}_m^j(t)$ where

$$\begin{pmatrix} dx_1^j \\ dx_2^j \\ \vdots \\ dx_m^j \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{R}\, L_2^j L_1^j x_0^j dt/2 \\ \vdots \\ \bar{R}\, L_m^j L_{m-1}^j x_{m-2}^j dt/2 \end{pmatrix} + \sqrt{R}\, \begin{pmatrix} L_1^j \\ L_2^j x_1^j \\ \vdots \\ L_m^j x_{m-1}^j \end{pmatrix} dw$$

and $x_0^j \equiv 1$, $x_i^j(0) = 0$ for $i > 0$.

Theorems 2 and 3 hold also, when

$$L(s_1,\ldots,s_m) = \sum_j \lambda_j \prod_{i=1}^{m} L_i^j(s_i)$$

and $\sum |\lambda_i| < \infty$ and the $\{L_i^j\}$ are uniformly bounded in $L_2[0,T]$ for each $T < \infty$. The extension of the last assertion of Theorem 2 requires boundedness of each $L_i^j(\cdot)$ for $i > 1$.

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